

Holomorphic maps with large images

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Dedicated to the memory of Shoshichi Kobayashi

Abstract

We show that each pseudoconvex domain in \mathbb{C}^n admits a dense set of holomorphic maps to \mathbb{C}^m whose sets of Casorati-Weierstrass points coincide with the whole boundary. As an application, we obtain that each irreducible complex space with a nonconstant holomorphic function admits a holomorphic map with dense image in \mathbb{C}^m , which leads to a new anti-hyperbolic concept called universally dominated complex spaces, together with a number of basic properties and examples. Analogous results are obtained for Plessner points and Picard points. We also study complex manifolds which admits a holomorphic map onto some \mathbb{C}^m . In particular, we show that each bounded domain with Lipschitz boundary in \mathbb{C}^n admits a locally biholomorphic map onto \mathbb{C}^n .

Mathematics Subject Classification (2000): 32H02, 32H35, 32T05.

Keywords: Holomorphic map, Casorati-Weierstrass point, Plessner point, Picard point, universally dominated space.

1 Introduction

We start with the following

Casorati-Weierstrass Theorem. *A point $p \in \mathbb{C}$ is an essential singularity of a holomorphic function f in a deleted neighborhood of p if and only if for each $\zeta \in \mathbb{C}_\infty$, the extended complex plane, there is a sequence $\{z^j\}_{j=1}^\infty$ of points converging to p such that $f(z^j) \rightarrow \zeta$ as $j \rightarrow \infty$.*

Picard's Great Theorem. *In each neighborhood of its isolated essential singularity, a holomorphic function f assumes each complex number, with one possible exception, an infinite number of times.*

These classical theorems mark the beginning of two great theories: the theory of cluster sets and the value distribution theory of R. Nevanlinna. The present work is somewhat close to the former. Following [9], we introduce the following

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Definition. Consider a domain $\Omega \subset \mathbb{C}^n$ and a holomorphic map $F : \Omega \rightarrow \mathbb{C}^m$.

(i) A point $p \in \partial\Omega$ is said to be a *Casorati-Weierstrass point* of F if the cluster set $C_\Omega(F, p)$ of F at p coincides with the Osgood space $\mathbb{C}_\infty^m := (\mathbb{C}_\infty)^m$, where

$$C_\Omega(F, p) = \bigcap_{r>0} \overline{F(\Omega \cap B(p, r))},$$

$B(p, r)$: being the ball with center p and radius r .

(ii) A point $p \in \partial\Omega$ is said to be a *Picard point* of F if $m = 1$ and F assumes infinitely often in any neighborhood of p all complex numbers with at most one exception.

Suppose furthermore that $\partial\Omega$ is C^2 .

(iii) A point $p \in \partial\Omega$ is said to be a *Plessner point* of F if each non-tangential conical cluster set $C_\Lambda(F, p)$ coincides with \mathbb{C}_∞^m where

$$C_\Lambda(F, p) = \bigcap_{r>0} \overline{F(\Omega \cap B(p, r) \cap \Lambda)}$$

Λ : being a non-tangential (finite) cone with vertex at p , i.e., it is contained in some $\Gamma_\alpha(p)$ where

$$\Gamma_\alpha(p) = \{z \in \Omega : |z - p| < (1 + \alpha)\delta_\Omega(z)\}, \quad \alpha > 0.$$

Let $\mathcal{O}(\Omega, \mathbb{C}^m)$ be the space of all holomorphic maps $F : \Omega \rightarrow \mathbb{C}^m$ with the topology of uniform convergence on compact subsets. Let $\mathcal{CW}(\Omega, \mathbb{C}^m) \subset \mathcal{O}(\Omega, \mathbb{C}^m)$ denote the set of all holomorphic maps whose sets of Casorati-Weierstrass points coincide with $\partial\Omega$. Analogously we may define $\mathcal{PL}(\Omega, \mathbb{C}^m)$ for Plessner points when $\partial\Omega$ is C^2 .

Theorem 1.1. *If $\Omega \subset \mathbb{C}^n$ is a pseudoconvex domain, then $\mathcal{CW}(\Omega, \mathbb{C}^m)$ is dense in $\mathcal{O}(\Omega, \mathbb{C}^m)$ for any $m \geq 1$. Suppose furthermore that $\partial\Omega$ is C^2 , then $\mathcal{PL}(\Omega, \mathbb{C}^m)$ is also dense in $\mathcal{O}(\Omega, \mathbb{C}^m)$.*

Let $\mathcal{P}(\Omega)$ denote the set of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ whose sets of Picard points coincide with $\partial\Omega$.

Theorem 1.2. *If $\Omega \subset \mathbb{C}^n$ is a pseudoconvex domain, then $\mathcal{P}(\Omega)$ is dense in $\mathcal{O}(\Omega)$.*

Sometimes it is interesting to ask for a function lying in certain classical function spaces with a point $p \in \partial\Omega$ as Picard point. Recall that the Bergman space is defined by

$$A_\alpha^2(\Omega) := \left\{ f \in \mathcal{O}(\Omega) : \int_\Omega |f|^2 \delta_\Omega^\alpha < \infty \right\}, \quad \alpha > -1.$$

(We also write $A^2(\Omega)$ for $A_0^2(\Omega)$ for the sake of simplicity).

Theorem 1.3. *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with $n \geq 2$. Let $p \in \partial\Omega$ be a point which admits an inner ball, i.e., a ball contained in Ω whose boundary intersects $\partial\Omega$ only at p . Then for each $\alpha > 0$ there is a function in $A_\alpha^2(\Omega)$ with p as Picard point. Suppose furthermore there is a negative plurisubharmonic (psh) function ψ on Ω satisfying $C_1|z - p|^{\beta_1} \leq -\psi(z) \leq C_2|z - p|^{\beta_2}$ for some positive constants $C_1, C_2, \beta_1, \beta_2$, then there is a holomorphic function in $A^2(\Omega)$ with p as Picard point.*

We remark that the first conclusion fails when $n = 1$ or $\alpha = 0$. To see this, consider simply the domain $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ or $\mathbb{D} \times \mathbb{D}^*$, where \mathbb{D} is the unit disc in \mathbb{C} . If $f \in A_\alpha^2(\mathbb{D}^*)$ with $\alpha > 0$, then $z = 0$ would be either a removable singularity or a pole of f , which is not a Picard point. If $f \in A^2(\mathbb{D} \times \mathbb{D}^*)$, then $f \in A^2(\mathbb{D}^2)$ so that the origin is not a Picard point for f . Nevertheless, there is an inner ball at the origin for both cases. Note also that the condition in the second conclusion holds when Ω is of finite type in the sense of D'Angelo (cf. [6]).

Remark. These types of boundary points are useful in the study of proper maps from discs to \mathbb{C}^2 (cf. [14]).

As applications of these theorems, we obtain some results concerning holomorphic maps with large images:

- (1) *An irreducible complex space admits a holomorphic map to \mathbb{C}^m with dense image if and only if it admits a nonconstant holomorphic function.*
- (2) *An irreducible complex space admits a holomorphic map onto \mathbb{C} if and only if it admits a nonconstant holomorphic function.*
- (3) *Every noncompact Riemann surface admits a holomorphic map onto \mathbb{C} which is locally biholomorphic; Every Riemann surface admits a holomorphic map onto \mathbb{P}^1 .*
- (4) *Every bounded domain with Lipschitz boundary or bounded homogeneous domain in \mathbb{C}^n admits a holomorphic map onto \mathbb{C}^n which is locally biholomorphic.*
- (5) *The complement of any quasi-ample analytic hypersurface in a projective algebraic manifold of dimension n admits a non-degenerate holomorphic map onto \mathbb{C}^n .*

We remark some results closely related to our work. First of all, Fornaess-Stout proved that the unit ball or polydisc admits a locally biholomorphic, finite holomorphic map onto any complex manifold of same dimension (cf. [11], [12]). Slightly later, Low [31] showed that each bounded domain with C^2 -boundary in \mathbb{C}^n admits a holomorphic map onto \mathbb{C}^n . His map is neither locally biholomorphic nor finite, however. Recently, Winkelmann [41] and Forstnerič-Winkelmann [17] proved the following

- (a) *An irreducible complex space is universal dominating, i.e., it admits a holomorphic map to any irreducible complex space with dense image, if and only if it admits a nonconstant bounded holomorphic function.*
- (b) *Every holomorphic map from \mathbb{C} to \mathbb{C}^n can be approximated uniformly on compact subsets by a sequence of holomorphic maps from \mathbb{C} to \mathbb{C}^n with dense image.*
- (c) *For any complex manifold X , the set of all holomorphic maps from the unit disc \mathbb{D} to X with dense images lies dense in $\mathcal{O}(\mathbb{D}, X)$.*

Remark. Indeed, (c) is a special case of the Oka principle in Section 5.4 of Forstnerič's book [13].

Inspired by these facts, we would like to introduce the following

Definition. A complex space is said to be *universally dominated* if it is dominated by any irreducible complex space with a nonconstant holomorphic function.

Recall that a complex space M is said to be dominated by another complex space M' if there is a dominant morphism $F : M' \rightarrow M$, i.e., a holomorphic map with dense image. The point is that the pullback of F defines an injective homomorphism $F^* : \mathcal{M}(M) \rightarrow \mathcal{M}(M')$

between fields of meromorphic functions on M and M' , respectively (compare [27]). It follows that a universally dominated space M has "less" meromorphic functions than any irreducible complex space M' with $\mathcal{O}(M') \neq \mathbb{C}$.

We infer from (1) that a complex space is universally dominated if and only if it is dominated by some \mathbb{C}^m . Thus this class of complex spaces has in fact been considered by several authors in certain special categories (see e.g., [4], [41]). In this paper, we obtain some basic properties of universally dominated spaces, together with a number of examples and problems. In particular, universal dominatedness is stable under *birational* transformations (see Corollary 5.17).

One warning: the notion of "universally dominated" is different from the standard notion of dominability where one requires the existence of a map with surjective derivative at some point.

2 Holomorphic maps with dense images in \mathbb{C}^m

Let Ω be a pseudoconvex domain in \mathbb{C}^n and K a compact subset of Ω . We take a C^∞ strictly psh exhaustion function ρ on Ω such that $K \subset \{\rho < 0\}$. The following approximation-interpolation result will play a central role in this paper.

Lemma 2.1. *Let $\{z^j\}_{j=1}^\infty$ be a discrete sequence of points in $\{\rho \geq 2\}$. Let $F_0 \in \mathcal{O}(\Omega, \mathbb{C}^m)$, and $\{c^j\}_{j=1}^\infty$ be a sequence of points in \mathbb{C}^m . Then for any $\varepsilon > 0$ there is a map $F \in \mathcal{O}(\Omega, \mathbb{C}^m)$ such that $F(z^j) = c^j$ for each j and $|F - F_0|_K := \sup_K |F - F_0| < \varepsilon$.*

Proof. For each j , we define

$$r_j = \min \left\{ \frac{1}{4} \min \{ |z^k - z^j| : k \neq j \}, \min \{ |z - z^j| : \rho(z) \leq 1 \} \right\}.$$

Let $0 \leq \chi \leq 1$ be a C^∞ function such that $\chi|_{(-\infty, 1/2)} = 1$ and $\chi|_{(1, \infty)} = 0$. We may choose a convex increasing function λ on \mathbb{R} such that $\lambda = 0$ on $(-\infty, 0]$ and

$$1) i\partial\bar{\partial}(\lambda \circ \rho + 2n \sum_j \chi(|z - z^j|/r_j) \log(|z - z^j|/r_j)) \geq i\partial\bar{\partial}|z|^2 + i\partial\rho \wedge \bar{\partial}\rho \text{ on } \{\rho \geq 1/2\}.$$

$$2) \sum_{j=0}^\infty b'_j < \varepsilon, \text{ where } b'_0 = \int_{\{1/2 \leq \rho \leq 1\}} |F_0|^2 e^{-\lambda \circ \rho} \text{ and } b'_j = (|c_j|/r_j)^2 \int_{B(z^j, r_j)} e^{-\lambda \circ \rho} \text{ for } j \geq 1.$$

Put

$$\varphi(z) = \lambda \circ \rho(z) + 2n \sum_{j=1}^\infty \chi(|z - z^j|/r_j) \log(|z - z^j|/r_j)$$

and

$$v = (v_1, \dots, v_m) := F_0 \bar{\partial}(\chi \circ \rho) + \sum_{j=1}^\infty c^j \bar{\partial}(\chi(|z - z^j|/r_j)).$$

Clearly, v is a C^∞ $\bar{\partial}$ -closed m -vector valued $(0, 1)$ form on Ω and by 1), 2),

$$\int_\Omega (|v_1|_{i\partial\bar{\partial}\varphi}^2 + \dots + |v_m|_{i\partial\bar{\partial}\varphi}^2) e^{-\varphi} \leq C \sum_{j=0}^\infty b'_j < C\varepsilon$$

for suitable constant $C > 0$. Applying Hörmander's L^2 -estimates for the $\bar{\partial}$ -equation (with values in the trivial m -vector bundle) (cf. [28], see also [1], [10]), we may solve the equation $\bar{\partial}u = v$ on Ω together with estimate

$$\int_{\Omega} |u|^2 e^{-\varphi} \leq \int_{\Omega} (|v_1|_{i\bar{\partial}\bar{\partial}\varphi}^2 + \cdots + |v_m|_{i\bar{\partial}\bar{\partial}\varphi}^2) e^{-\varphi} < C\varepsilon.$$

It suffices to take

$$F(z) = (\chi \circ \rho)F_0(z) + \sum_{j=1}^{\infty} c^j \chi(|z - z^j|/r_j) - u(z).$$

Q.E.D.

Proof of Theorem 1.1. For each positive integer μ , we define

$$K_{\mu} = \{z \in \Omega : 2^{-2\mu} \leq \delta_{\Omega}(z) \leq 2^{-2\mu+1}\} \cap \overline{B(0, \mu)}.$$

Take a finite number of points $z^{\mu_1}, \dots, z^{\mu_{k_{\mu}}}$ in K_{μ} such that the balls $\{B(z^{\mu_s}, 2^{-2\mu-2})\}_{1 \leq s \leq k_{\mu}}$ cover K_{μ} . Pick $\mu_0 > 0$ such that $K_{\mu} \cap \{\rho \leq 2\} = \emptyset$ for each $\mu \geq \mu_0$. The set $\{z^{\mu_s} : 1 \leq s \leq k_{\mu}, \mu > \mu_0\}$ is countable and discrete in Ω , which we denote by $\{z^j\}_{j=1}^{\infty}$ for the sake of simplicity.

Now define by induction a sequence $\{n_{\mu, \nu} : \mu, \nu \geq 1\}$ of positive integers with two indices as follows. We renumerate the sequence of positive even numbers by $\{n_{\mu, 1}\}_{\mu=1}^{\infty}$. Suppose we have chosen $\{n_{\mu, k-1}\}_{\mu=1}^{\infty}$. Then $\{n_{\mu, k}\}_{\mu=1}^{\infty}$ would be a renumeration of the sequence $\{n_{2\mu, k-1}\}_{\mu=1}^{\infty}$. Let $\{\zeta^k\}_{k=1}^{\infty}$ be a dense sequence of points in \mathbb{C}^m . We define a sequence $\{c^j\}_{j=1}^{\infty}$ of points in \mathbb{C}^m by $c^j = \zeta^k$ for all j with $z^j \in K_{n_{2\mu-1, k}}$, and $c^j = 0$ otherwise. By virtue of Lemma 2.1, we have a holomorphic map $F : \Omega \rightarrow \mathbb{C}^m$ such that $F(z^j) = c^j$ for each j and $|F - F_0|_K < \varepsilon$. By the construction of $\{z^j\}_{j=1}^{\infty}$, we conclude that for each k the set of cluster points of $\{z^j \in K_{n_{2\mu-1, k}} : \mu \geq 1\}$ contains the whole boundary $\partial\Omega$. Thus the sequence $\{\zeta^k\}_{k=1}^{\infty}$ is contained in $C_{\Omega}(F, p)$ for any $p \in \partial\Omega$, so is the whole of \mathbb{C}_{∞}^m .

The second assertion may be proved analogously with minor modifications. We take for each μ a finite number of points $z^{\mu_1}, \dots, z^{\mu_{k_{\mu}}}$ in K_{μ} such that the balls $\{B(z^{\mu_s}, 2^{-4\mu})\}_{1 \leq s \leq k_{\mu}}$ cover K_{μ} . Given a point $p \in \partial\Omega$ and a non-tangential cone Λ with vertex at p , we pick a point p_{μ} in the intersection of K_{μ} with the axis of Λ for sufficiently large μ . It is easy to see that $|p_{\mu} - p| \asymp 2^{-2\mu}$. Thus there is a number $\varepsilon > 0$ such that $B(p_{\mu}, \varepsilon 2^{-2\mu}) \subset \Lambda$ for all sufficiently large μ . Since $\{B(z^{\mu_s}, 2^{-4\mu})\}_{1 \leq s \leq k_{\mu}}$ cover K_{μ} , we conclude that for each sufficiently large μ the ball $B(p_{\mu}, \varepsilon 2^{-2\mu})$ contains at least one z^{μ_s} , so does Λ . The remaining argument is completely same as above. Q.E.D.

Remark. It remains open whether the second assertion in Theorem 1.1 holds for each radial cluster set defined by

$$C_l(F, p) = \bigcap_{r>0} \overline{F(\Omega \cap B(p, r) \cap \mathbb{R}v_p)},$$

$\mathbb{R}v_p$: being the real normal at p . On the other hand, Globevnik-Stout [23] has shown that for each strongly pseudoconvex domain Ω there exists a holomorphic function f such that $\lim_{t \rightarrow 1^-} f(\gamma(t))$ does not exist for any smooth curve $\gamma : [0, 1) \rightarrow \Omega$ with $\gamma(1) \in \partial\Omega$ and $\gamma'(1)$ not tangent to $\partial\Omega$ at $\gamma(1)$.

As a consequence of Theorem 1.1, we immediately get the following

Proposition 2.2. *An irreducible complex space admits a holomorphic map to \mathbb{C}^m with dense image if and only if it admits a nonconstant holomorphic function.*

Proof. It suffices to verify the if part. Since every nonconstant holomorphic function on an irreducible complex space defines an open map to \mathbb{C} , we only need to construct a holomorphic map from an open set in \mathbb{C} to \mathbb{C}^m with dense image. But each open set in \mathbb{C} is pseudoconvex, Theorem 1.1 applies. Q.E.D.

We also have the following analogous result in real-analytic category:

Proposition 2.3. *Each noncompact real-analytic manifold admits a real-analytic map to \mathbb{R}^m with dense image.*

Proof. By virtue of Grauert's theorem [19], each real-analytic manifold M admits a Stein neighborhood U in the total space of the tangent bundle TM of M , with respect to the natural complexification. Suppose now M is noncompact, then there is a discrete sequence $\{q^k\}_{k=1}^\infty$ of points in M . Let $\{\zeta^j\}_{j=1}^\infty$ be a dense sequence of points in $\mathbb{R}^m \subset \mathbb{C}^m$. By virtue of Lemma 2.1, one can construct a holomorphic map $F : U \rightarrow \mathbb{C}^m$ such that $F(q^{k_j}) = \zeta^j$ for some subsequence $\{q^{k_j}\}$. The restriction of F to M gives the desired real-analytic map. Q.E.D.

3 Picard points

Proofs of Theorem 1.2 and 1.3 rely on several types of Lindelöf principles. We first recall the following classical

Proposition 3.1 (cf. [39], p. 308). *Let D be a domain in \mathbb{C} , which is bounded by a Jordan curve C and $z = 0$ belong to C . Then by $z = 0$, the part of C which lies in a neighborhood of $z = 0$ is decomposed into two parts C_1, C_2 . If f is a holomorphic function on D satisfying $\lim_{z \rightarrow 0} f(z) = a$, when $z \rightarrow 0$ on C_1 and $\lim_{z \rightarrow 0} f(z) = b$, when $z \rightarrow 0$ on C_2 and $a \neq b$, then f takes any value infinitely often in D , with one possible exception.*

As a consequence, we obtain the following lemma which will be used in the subsequent section.

Lemma 3.2. *Let Ω be a domain in \mathbb{C} and $p \in \partial\Omega$. Suppose Ω contains a cone Λ_p with vertex at p . Then there is a holomorphic function f on $\mathbb{C} \setminus \{p\}$ such that p is a Picard point of f on Ω and $f'(z) \neq 0$ for each $z \in \mathbb{C} \setminus \{p\}$.*

Proof. Put $f_1(z) = e^{1/z^2}$. Clearly, $f_1 \in \mathcal{O}(\mathbb{C}^*)$. Write $z = x + iy$. Since

$$f_1(z) = e^{\frac{x^2-y^2}{(x^2+y^2)^2}} e^{-\frac{2ixy}{(x^2+y^2)^2}},$$

we see that for each $0 < \varepsilon < 1$, $\lim_{z \rightarrow 0} f_1(z) = \infty$, when $z \rightarrow 0$ on the ray $l_1 : y = (1 - \varepsilon)x$, $x > 0$, and $\lim_{z \rightarrow 0} f_1(z) = 0$, when $z \rightarrow 0$ on the ray $l_2 : y = (1 + \varepsilon)x$, $x > 0$. By virtue of the previous proposition, we conclude that 0 is a Picard point for f_1 on the following (infinite) cone

$$V_\varepsilon = \{(x, y) \in \mathbb{R}^2 : (1 - \varepsilon)x < y < (1 + \varepsilon)x, x > 0\}.$$

For sufficiently small ε , we have a complex affine map $\Phi : V_\varepsilon \cap B(0, \varepsilon) \rightarrow \Lambda_p$ obtained by a composition of translation and rotation such that $\Phi(0) = p$. The desired function may be chosen as $f = f_1 \circ \Phi^{-1}$. Q.E.D.

Remark. The ray $y = x, x > 0$ is a Julia ray of f . Julia proved that each $f \in \mathcal{O}(\mathbb{C}^*)$ with 0 as essential singularity has at least one Julia ray.

Next we recall two types of Lindelöf principles of several complex variables as follows.

Definition. Let M, N be complex manifolds, and $\mathcal{O}(M, N)$ be the set of holomorphic maps from M to N . A map $F \in \mathcal{O}(M, N)$ is called *normal* if the family $\{F \circ H : H \in \mathcal{O}(\mathbb{D}, M)\}$ forms a normal family in the sense of Wu [42].

We remark that for a bounded domain $\Omega \subset \mathbb{C}^n$, a function $f \in \mathcal{O}(\Omega)$ is normal if it omits (at least) two values (cf. [8]).

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^2 -boundary. Let $f \in \mathcal{O}(\Omega)$ and $p \in \partial\Omega$. We say that f has non-tangential limit c at p if

$$\lim_{\Gamma_\alpha(p) \ni z \rightarrow p} f(z) = c$$

for any $\alpha > 0$, where

$$\Gamma_\alpha(p) = \{z \in \Omega : |z - p| < (1 + \alpha)\delta_\Omega(z)\}.$$

A curve $\gamma : [0, 1) \rightarrow \Omega$ which terminates at p is said to be non-tangential if it is contained in some $\Gamma_\alpha(p)$.

Proposition 3.3 (cf. [8], Lemma 3.1 and the subsequent remark). *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^2 -boundary and $p \in \partial\Omega$. Suppose $f \in \mathcal{O}(\Omega)$ is a normal function, and f has limit c along some non-tangential curve γ terminating at p . Then f has non-tangential limit c at p .*

Sometimes it is more interesting to find sufficient conditions on a sequence $\Omega \ni z^j \rightarrow p \in \partial\Omega$ for a normal function $f \in \mathcal{O}(\Omega)$ to have the non-tangential limit c . Generalizing the classical results of Bagemihl-Seidel [3] from one complex variable, K. T. Hahn proved the following

Proposition 3.4 (cf. [26], Theorem 4 and the subsequent remark). *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^2 -boundary at $p \in \partial\Omega$. Let $\{z^j\}$ be a sequence of points in Ω which tends to p such that*

- (a) $\lim_{j \rightarrow \infty} k_\Omega(z^j, z^{j+1}) = 0,$
- (b) $\lim_{j \rightarrow \infty} \frac{d^2(z^j, \mathbb{C}v_p)}{d(z^j, \mathbb{C}T_p)} = 0.$

If $f \in \mathcal{O}(\Omega)$ is a normal function such that $f(z^j) \rightarrow c$ as $j \rightarrow \infty$, then f has non-tangential limit c at p . Here $\mathbb{C}T_p$ and $\mathbb{C}v_p$ are the complex tangent space and the complex normal space at p , respectively, and k_Ω is the Kobayashi distance of Ω .

Remark. Actually, Hahn's theorem is much more general. He considered normal maps from Ω to a relatively compact open set X in a hermitian manifold N with hermitian distance d_N . Note that \mathbb{C} may be regarded as a relatively compact domain in the Riemann sphere with Fubini-Study distance.

Proof of Theorem 1.2. We claim that there are a dense sequence $\{p^j\}$ of points in $\partial\Omega$ and a disjoint sequence of balls $\{B_j\}$ in Ω such that ∂B_j intersects $\partial\Omega$ only at p^j for each j . To see this, take first a dense sequence $\{q^j\}$ of points in $\partial\Omega$. If q^1 is an isolated point of $\partial\Omega$, then we take $p^1 = q^1$ and a small ball B_1 in Ω such that ∂B_1 intersects $\partial\Omega$ only at p^1 . Otherwise, we may choose $a^1 \in B(q^1, \frac{1}{2}) \cap \Omega$ and $p^1 \in \partial\Omega$ such that $|a^1 - p^1| = \delta_\Omega(a^1)$. It suffices to take

$$B_1 = B((a^1 + p^1)/2, \delta_\Omega(a^1)/2).$$

Suppose we have chosen p^j and B_j for $1 \leq j \leq k-1$. If q^k is an isolated point of $\partial\Omega$, then we take $p^k = q^k$ and a small ball B_k in $\Omega \setminus \bigcup_{j=1}^{k-1} \overline{B_j}$ such that ∂B_k intersects $\partial\Omega \cup \bigcup_{j=1}^{k-1} \partial B_j$ only at p^k . Otherwise, we may choose $a^k \in B(q^k, \frac{1}{2k}) \cap \Omega \setminus \bigcup_{j=1}^{k-1} \overline{B_j}$ and $p^k \in \partial\Omega$ such that $|a^k - p^k| = \delta_\Omega(a^k) < d(a^k, \bigcup_{j=1}^{k-1} \overline{B_j})$. It suffices to take $B_k = B(\frac{1}{2}(a^k + p^k), \frac{1}{2}\delta_\Omega(a^k))$.

Now choose in each B_j a sequence $\{z^{j\mu}\}_{\mu=1}^\infty$ of points on the radius terminating at p^j such that

$$\lim_{\mu \rightarrow \infty} z^{j\mu} = p^j \quad \text{and} \quad \lim_{\mu \rightarrow \infty} k_{B_j}(z^{j(2\mu-1)}, z^{j(2\mu+1)}) = 0.$$

Let K and ρ be as in the previous section. Without loss of generality, we may assume that $\{z^{j\mu} : j, \mu \geq 1\} \subset \{\rho \geq 2\}$. By virtue of Lemma 2.1, we have for each $f_0 \in \mathcal{O}(\Omega)$ and each $\varepsilon > 0$ a function $f \in \mathcal{O}(\Omega)$ such that $f(z^{j(2\mu-1)}) = 0$, $f(z^{j(2\mu)}) = 1$ for any $j, \mu \geq 1$, and $|f - f_0|_K < \varepsilon$. We claim that each p^j is a Picard point of f . Suppose on the contrary that p^j is not a Picard point, then there would be a neighborhood U_j of p^j such that $f|_{\Omega \cap U_j}$ omits at least two values, in particular, it is a normal function on $B_j \cap U_j$. Thus by virtue of Proposition 3.4, it follows that $f|_{B_j \cap U_j}$ has non-tangential limit 0 at p^j . Contradiction.

We conclude the proof by the following trivial lemma. Q.E.D.

Lemma 3.5. *Let Ω be a domain in \mathbb{C}^n and $f \in \mathcal{O}(\Omega)$. Suppose there is a dense set $\{p^j\} \subset \partial\Omega$ of Picard points for f , then $f \in \mathcal{P}(\Omega)$.*

Proof. Let $p \in \partial\Omega$ be given. Suppose p is not a Picard point for f , then there would be a number $r > 0$ such that $f(B(p, r) \cap \Omega)$ omits at least two complex numbers a, b . Since $\{p^j\}$ is dense in $\partial\Omega$, we have at least one point $p_{j_0} \in B(p, r/2)$. Thus $f(B(p_{j_0}, r/2) \cap \Omega)$ also omits a, b , so that p_{j_0} is not a Picard point of f . Contradiction. Q.E.D.

To prove Theorem 1.3, we need a general extension theorem of Ohsawa as follows.

Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain and let S be a closed complex submanifold of Ω such that each component of S is a domain in some complex affine subspace of \mathbb{C}^n . We denote by $\#(S)$ the set of all *negative* $\Psi \in PSH(\Omega)$ satisfying the following conditions:

- (i) $S \subset \Psi^{-1}(-\infty)$;
- (ii) If S is k -dimensional around a point x , there exists a local coordinates (z_1, \dots, z_n) on a neighborhood U of x such that $z_{k+1} = \dots = z_n = 0$ on $S \cap U$ and

$$\sup_{U \setminus S} \left| \Psi(z) - (n-k) \log \sum_{j=k+1}^n |z_j|^2 \right| < \infty.$$

Let dV denote the Lebesgue measure in \mathbb{C}^n . For each $\Psi \in \#(S)$, one can define a positive measure $dV[\Psi]$ on S as the minimum element of the partially ordered set of positive measure

$d\mu$ satisfying

$$\int_S f d\mu \geq \limsup_{t \rightarrow +\infty} \frac{2(n-k)}{\sigma_{2n-2k-1}} \int_{\Psi^{-1}((-t-1, -t))} f e^{-\Psi} dV$$

for any nonnegative continuous function f which is compactly supported in Ω . Here σ_m denotes the volume of the unit sphere in \mathbb{R}^{m+1} .

Theorem (cf. [36]). *Let Ω be a pseudoconvex domain in \mathbb{C}^n and let S be as above. Let φ be a psh function on Ω and $\Psi \in \#(S)$. Then for any holomorphic function f on S satisfying $\int_S |f|^2 e^{-\varphi} dV[\Psi] < +\infty$, there exists a holomorphic function F on Ω such that $F|_S = f$ and*

$$\int_{\Omega} |F|^2 e^{-\varphi} dV \leq 2^8 \pi \int_S |f|^2 e^{-\varphi} dV[\Psi].$$

Proof of Theorem 1.3. Let Ω_2 denote the intersection of Ω with a 2-dimensional complex affine subspace intersecting $\partial\Omega$ transversally at p (note that Ω admits an inner ball at p). By virtue of the Ohsawa-Takegoshi extension theorem [35], each function in $A_{\alpha}^2(\Omega_2)$ with $\alpha \geq 0$ extends to a function in $A_{\alpha}^2(\Omega)$. Thus it suffices to consider the case when $n = 2$. Without loss of generality, we may assume that $p = 0$ and the inner ball at 0 is $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1 - 1|^2 + |z_2|^2 < 1\}$. Put

$$L_1 = \{(z_1, z_2) \in \mathbb{C}^2 : z_2 = 0\}, \quad L_2 = \{(z_1, z_2) \in \mathbb{C}^2 : z_2 = z_1\},$$

and $S := S_1 \cup S_2$, where $S_k = \Omega \cap L_k$ for $k = 1, 2$. Let f be a holomorphic function on S given by $f = 1$ on S_1 and $f = 0$ on S_2 . Let d denote the diameter of Ω . With respect to the functions

$$\varphi(z) := \alpha \log 1/\delta_{\Omega}(z), \quad \Psi(z) := \log |z_2|^2 + \log |z_2 - z_1|^2 - 2 \log(2d^2) \in \#(S),$$

we have the estimate

$$\int_S |f|^2 e^{-\varphi} dV[\Psi] = \int_{S_1} e^{-\varphi} dV[\Psi] = 4d^4 \int_{S_1} |z_1|^{-2} \delta_{\Omega}^{\alpha} \leq 4d^4 \int_{\{z_1: 0 < |z_1| < d\}} |z_1|^{-2+\alpha} < \infty.$$

Thus by the previous extension theorem, there is a holomorphic extension F of f to Ω such that

$$\int_{\Omega} |F|^2 e^{-\varphi} dV < \infty,$$

i.e., $F \in A_{\alpha}^2(\Omega)$. Now $F = 1$ along the real line $l_1 := \{(x, 0) \in \mathbb{C}^2 : x \in \mathbb{R}\}$ and $F = 0$ along the real line $\{(x, x) \in \mathbb{C}^2 : x \in \mathbb{R}\}$, we see that 0 is a Picard point of F by virtue of Proposition 3.3.

For the second assertion, we put $\varphi = 0$ and

$$\Psi(z) := -\frac{4}{\beta_1} \log(-\psi) + \log |z_2|^2 + \log |z_2 - z_1|^2 - \frac{4}{\beta_1} \log C_1 - \log 4.$$

Since $-\psi(z) \geq C_1 |z|^{\beta_1}$, we conclude that $\Psi \in \#(S)$. For the function f defined as above, we have

$$\int_S |f|^2 dV[\Psi] = \int_{S_1} dV[\Psi] = C \int_{S_1} |z_1|^{-2} (-\psi)^{4/\beta_1} \leq C \int_{\{z_1: 0 < |z_1| < d\}} |z_1|^{-2+4\beta_2/\beta_1} < \infty$$

since $-\psi(z) \leq C_2 |z|^{\beta_2}$. Applying the extension theorem again, we see that f admits a holomorphic extension $F \in A^2(\Omega)$. By a similar argument as above, we conclude that 0 is a Picard point of F . Q.E.D.

4 Holomorphic maps onto \mathbb{C}^n

We begin with the following

Proposition 4.1. *An irreducible complex space admits a holomorphic map onto \mathbb{C} if and only if it admits a nonconstant holomorphic function.*

Proof. It suffices to verify the if part. Let h be a nonconstant holomorphic function on an irreducible complex space M . Then the image $h(M)$ is an open set in \mathbb{C} . Thanks to Theorem 1.3, there is a surjective holomorphic map $g : h(M) \rightarrow \mathbb{C}^*$. On the other side, the holomorphic function $f(z) = z + \frac{1}{z}$ maps \mathbb{C}^* onto \mathbb{C} . Clearly, $f \circ g \circ h$ defines a surjective holomorphic map from M to \mathbb{C} . Q.E.D.

In order to study locally biholomorphic maps, we need the following

Lemma 4.2. *There is a surjective holomorphic map $f : \mathbb{C}^* \rightarrow \mathbb{C}$ which is locally biholomorphic.*

Proof. It suffices to find a non-polynomial entire function f which maps \mathbb{C} locally biholomorphically onto \mathbb{C} : since ∞ is an essential singularity of f , there exists $a \in \mathbb{C}$ whose preimage contains at least two points, say 0, 1 for the sake of simplicity. It follows that $f|_{\mathbb{C}^*}$ is a desired holomorphic map. Following E. Calabi (see [38], p. 640, or [43], p. 135), we may simply take

$$f(z) = \int_0^z e^{-w^2} dw.$$

Q.E.D.

Proposition 4.3. *Every noncompact Riemann surface admits a holomorphic map onto \mathbb{C} which is locally biholomorphic.*

Proof. Let M be a noncompact Riemann surface. By virtue of Gunning and Narasimhan's theorem (cf. [25]), there is a function $h \in \mathcal{O}(M)$ without critical points. Its image $h(M)$ is a domain in \mathbb{C} . We only need to consider the case $h(M) \neq \mathbb{C}$. Since $h(M)$ has at least one boundary point which admits an inner ball, we infer from Lemma 3.2 that there is a surjective holomorphic map $f : h(M) \rightarrow \mathbb{C}^*$ without critical points. Thus $f \circ h$ defines a holomorphic map from M onto \mathbb{C}^* , which is locally biholomorphic. The assertion follows immediately from the previous lemma. Q.E.D.

Corollary 4.4. *Every Riemann surface admits a holomorphic map onto \mathbb{P}^1 .*

Proof. It is well-known that each compact Riemann surface admits a holomorphic map onto \mathbb{P}^1 . Thus we only need to consider noncompact Riemann surfaces. The desired holomorphic map may be obtained by composing a holomorphic map onto \mathbb{C} , the universal covering map from \mathbb{C} to a torus, and a surjective holomorphic map from this torus to \mathbb{P}^1 . Q.E.D.

Proposition 4.5. *Let Ω be a domain in \mathbb{C}^n . Suppose there is a point $p \in \partial\Omega$ such that Ω contains a cone Λ_p with vertex at p , and there is a supporting complex hyperplane of Ω at p . Then there is a holomorphic map from Ω onto \mathbb{C}^n which is locally biholomorphic.*

Proof. After a change of (global) coordinate by a complex affine transformation, we may assume that $p = 0$, $\Omega \subset \{z \in \mathbb{C}^n : z_n \neq 0\}$, and the axis of Λ_p is contained in the complex line

$L := \{z_1 = \cdots = z_{n-1} = 0\}$. Thanks to Lemma 3.2, we have a holomorphic function $h(z_n)$ on $\mathbb{C}_{z_n}^*$ such that $h' \neq 0$ everywhere, and 0 is Picard point for h on the planar domain $\Lambda_p \cap L$. Now we define a holomorphic map $F = (f_1, \dots, f_n) : \Omega \rightarrow \mathbb{C}^n$ as follows

$$f_n(z) = h(z_n), \quad f_k(z) = z_k/z_n^2, \quad 1 \leq k \leq n-1.$$

Let $\zeta = (\zeta_1, \dots, \zeta_n) \in (\mathbb{C}^*)^n$ be arbitrarily fixed. By virtue of Lemma 3.2, we have a solution z_n^* to the equation $h(z_n) = \zeta_n$ which can be arbitrarily close to $z_n = 0$ inside the planar domain $\Lambda'_p \cap L$ where $\Lambda'_p \subset \Lambda_p$ is a smaller cone with the same axis. Put $z_k^* = \zeta_k(z_n^*)^2$ for each $1 \leq k \leq n-1$, and $z^* = (z_1^*, \dots, z_n^*)$. Clearly, we have $F(z^*) = \zeta$ and $z^* \in \Lambda_p$ provided $|z_n^*|$ sufficiently small. Thus F maps Ω onto $(\mathbb{C}^*)^n$. It is trivial to see that F is locally biholomorphic. Combining this with Lemma 4.2, we conclude the proof. Q.E.D.

Proposition 4.6. *A domain $\Omega \subset \mathbb{C}^n$ admits a holomorphic map onto \mathbb{C}^n which is locally biholomorphic if Ω belongs to one of the following domains:*

- (1) *bounded domains with Lipschitz boundaries.*
- (2) *convex domains.*
- (3) *bounded homogeneous domains.*
- (4) *model domains defined by*

$$\{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \text{Im } w > \psi(z)\}$$

where $\psi \geq 0$ is continuous function on \mathbb{C}^n so that $\psi(z) = O(|z|)$ holds near $z = 0$.

- (5) *Products $\Omega_1 \times \Omega_2$, where Ω_1 is a domain in \mathbb{C} and Ω_2 is a domain in \mathbb{C}^{n-1} .*

Proof. In case Ω is a bounded domain with Lipschitz boundary, we simply take the boundary point which is farthest from the origin so that the condition of the previous proposition is verified. Case (2) and (4) follow immediately from the previous proposition. Since each bounded homogeneous domain is biholomorphically equivalent to a Siegel domain of the second kind (cf. [40]), which is a convex (unbounded) domain, thus it admits a locally biholomorphic map onto \mathbb{C}^n . For the last case, we may assume that $\Omega_1 \subset \mathbb{C}^*$ (note that the universal covering map $\pi : \mathbb{C} \rightarrow \mathbb{C}^*$ is surjective and locally biholomorphic). Take a point $p_1 \in \partial\Omega_1$ so that Ω_1 admits an inner ball at p_1 . Without loss of generality, we may assume that the origin $0' \in \Omega_2$. Let f_1 be the holomorphic function given in Lemma 3.2. By an argument similar to the proof of the pervious proposition, we may take the desired holomorphic map from Ω onto $\mathbb{C}^* \times \mathbb{C}^{n-1}$ by

$$(z_1, z_2, \dots, z_n) \mapsto (f_1(z_1), z_2/(z_1 - p_1)^2, \dots, z_n/(z_1 - p_1)^2).$$

Combining this with Lemma 4.2, we conclude the proof. Q.E.D.

Conjecture 4.7. Every domain in \mathbb{C}^n , $n \geq 2$, admits a holomorphic map onto \mathbb{C}^n which is locally biholomorphic.

From the viewpoint of algebraic geometry, the most important noncompact complex manifolds are those Zariski open sets in a projective algebraic manifold. Let $S = \bigcup_{j=1}^N S_j$ be an analytic hypersurface in some projective algebraic manifold M with S_j being irreducible. S is said to be *quasi-ample* if there exist positive integers b_1, \dots, b_N , such that the effective Weil divisor $S_b = \sum_{j=1}^N b_j S_j$ is ample.

Proposition 4.8. *Let M be a projective algebraic manifold of dimension n and S be a quasi-ample analytic hypersurface in M . Then $M \setminus S$ admits a nondegenerate holomorphic map onto \mathbb{C}^n .*

Proof. Let $L = [S_b]$ be the ample line bundle over M associated to S_b (multiplied by a sufficient large positive integer, we may assume that L is very ample). Let s be a global holomorphic section of L whose associated divisor is precisely S_b . By Bertini's theorem, there are holomorphic sections $t_j \in H^0(M, L) \setminus \{0\}$, $1 \leq j \leq n-1$, such that the associated hypersurfaces $T_j = \{t_j = 0\}$ are smooth and $T_1, T_2, \dots, T_{n-1}, S$ are normal crossing at some smooth point p of S . Let R be the compact Riemann surface obtained by intersection of T_j , $1 \leq j \leq n-1$ and put $R^* = R \setminus \{p\}$. Since R^* is an open Riemann surface (hence is Stein), we may choose a holomorphic function f on R^* such that it equals 0 on a discrete sequence of points converging to p and equals to 1 on another discrete sequence of points converging to p by virtue of Cartan's Theorem A. Furthermore, we may choose f so that $f' \neq 0$ at some point which is sufficiently close to p . Thus p is an essential singularity of f . Since S_b is ample, $-\log |s|$ is a strictly plurisubharmonic exhaustion function of $M \setminus S$, we see that $M \setminus S$ is a Stein manifold. Thus f may be extended to a holomorphic function \tilde{f} on $M \setminus S$. We define a holomorphic map $F = (f_1, f_2, \dots, f_n) : M \setminus S \rightarrow \mathbb{C}^n$ by

$$f_n = \tilde{f}, \quad f_j = t_j/s, \quad 1 \leq j \leq n-1.$$

Since f omits at most one value, say 0, in each deleted neighborhood of p in R , we conclude that F maps $M \setminus S$ onto $\mathbb{C}^* \times \mathbb{C}^{n-1}$. Furthermore, the complex Jacobian of F is not identically zero. Combining this fact with Lemma 4.2, we conclude the proof. Q.E.D.

Corollary 4.9. *Let M be an Abelian variety and S an ample divisor of M . Then there is a surjective nondegenerate holomorphic map from $M \setminus S$ to M .*

Proof. Let $\pi : \mathbb{C}^n \rightarrow M$ be the universal covering map. By the previous proposition, there is a surjective nondegenerate holomorphic map $F : M \setminus S \rightarrow \mathbb{C}^n$. It suffices to take $\pi \circ F$. Q.E.D.

5 Universally dominated spaces

By virtue of Proposition 2.2, we immediately get

Proposition 5.1. *A complex space is universally dominated if and only if it is dominated by some \mathbb{C}^m .*

Basing on this fact, we obtain the following

Proposition 5.2. *Suppose M is universally dominated. Then we have*

- (1) *The Kobayashi pseudodistance k_M of M vanishes identically.*
- (2) *M is ultra-Liouville, i.e., any negative continuous psh function on M is constant.*
- (3) *If M is a projective algebraic manifold, then the irregularity of M , i.e., the dimension of the vector space of holomorphic 1-forms on M , is no greater than the dimension of M .*
- (4) *If M is a domain in \mathbb{C}^n , then for any complex line L , $\pi_L(M)$ omits at most one point in L where π_L is the projection from \mathbb{C}^n to L .*

Proof. (1) Take a dominant morphism $F : \mathbb{C} \rightarrow M$ (i.e., a holomorphic map with dense image). Given two points $z, w \in \mathbb{C}$, we have

$$k_M(F(z), F(w)) \leq k_{\mathbb{C}}(z, w) = 0.$$

Thus k_M vanishes on a dense set of $M \times M$, so that k_M has to vanish on $M \times M$ by continuity.

(2) Suppose on the contrary that there exists a negative nonconstant continuous psh function ψ on M . Let $F : \mathbb{C} \rightarrow M$ be a dominant morphism. Since ψ is continuous, it is nonconstant on $F(\mathbb{C})$. Thus $\psi \circ F$ would be a nonconstant negative subharmonic function on \mathbb{C} , which is absurd.

(3) Suppose on the contrary that the irregularity of M is greater than the dimension of M . By Bloch's theorem (cf. [34], [33], [22], [29]), we know that every holomorphic map $F : \mathbb{C} \rightarrow M$ has its image in a closed proper subvariety of M . Thus M is not universally dominated. Contradiction.

(4) Since π_L is an open map, $\pi_L(M)$ is an open set in L . If $L \setminus \pi_L(M)$ contains at least two points, then $\pi_L(M)$ is Kobayashi hyperbolic, so that k_M does not vanish. By (1), M could not be universally dominated. Contradiction. Q.E.D.

Proposition 5.3.

(1) Let M_1, M_2 be two complex spaces. The product $M_1 \times M_2$ is universally dominated if and only if both M_1, M_2 are universally dominated.

(2) Let M_1 be a universally dominated space. If M_2 is dominated by M_1 , then it is also universally dominated.

Proof. (1) The if part: take two dominant morphisms $F_j : \mathbb{C} \rightarrow M_j$. It follows immediately that $(F_1, F_2) : \mathbb{C}^2 \rightarrow M_1 \times M_2$ is dominant. The only if part: take a dominant morphism $\mathbb{C} \rightarrow M_1 \times M_2$. By composing with the projections $\pi_j : M_1 \times M_2 \rightarrow M_j$, $j = 1, 2$, respectively, we get dominant morphisms $\mathbb{C} \rightarrow M_j$.

(2) Take first a dominant morphism from \mathbb{C} to M_1 . By composing with a dominant morphism $M_1 \rightarrow M_2$, we get a dominant morphism $\mathbb{C} \rightarrow M_2$. Q.E.D.

Now we give some examples of universally dominated spaces.

Example 5.4. The Riemann surfaces which are universally dominated are $\mathbb{C}, \mathbb{C}^*, \mathbb{P}^1$ and all tori. Furthermore, any (singular) complex curve dominated by \mathbb{C} or \mathbb{P}^1 is universally dominated, e.g., the rational normal curve in \mathbb{P}^n , which is defined to be the image of the holomorphic map $\mathbb{P}^1 \rightarrow \mathbb{P}^n$ given by

$$(z_0 : z_1) \mapsto (z_0^n : z_0^{n-1} z_1 : \cdots : z_0 z_1^{n-1} : z_1^n).$$

Example 5.5. Elliptic $K3$ -surfaces and Kummer surfaces (cf. [4]).

Example 5.6. Fatou-Bieberbach domains, and unbounded domains $\mathbb{C}^n \setminus K$, where K is a strictly convex compact set in \mathbb{C}^n , $n \geq 2$ (cf. [37]).

Proposition 5.7. *There are domains $\Omega_1 \subset \Omega_2 \subset \mathbb{C}^n$ with $n \geq 2$ such that Ω_1 is universally dominated while Ω_2 is not.*

Proof. We simply take $\Omega_1 := \mathbb{C}^n \setminus \mathbb{B}^n$. Then it is universally dominated by virtue of the previous example. Fix a point p in the unit sphere and put $\Omega_2 = \Omega_1 \cup B(p, 1)$. We claim that

Ω_2 is not universally dominated. To see this, simply take a continuous psh peak function of Ω_2 at the strongly pseudoconvex point 0, so that Ω_2 is not universally dominated by virtue of Proposition 5.2/(2). Q.E.D.

Example 5.8. Toric spaces, i.e., complex spaces with an open dense subset biholomorphic to $(\mathbb{C}^*)^n$. One warning: this definition is more general than the classical definition of toric varieties in algebraic geometry). Classical examples of compact toric manifolds are the projective space \mathbb{P}^n , the Osgood space \mathbb{C}_∞^n , and the Hirzebruch surfaces. Moreover, the Grassmannians are also toric manifolds. To see this, simply note that every Grassmannian $\mathbb{G}(k, n)$ contains $\mathbb{C}^{k(n-k)}$ as a Zariski open subset (see e.g., [7], p. 320–321).

An important class of noncompact toric manifolds may be constructed as follow. Let S be a closed subset in \mathbb{C}^n so that there exists an automorphism F of \mathbb{C}^n such that $F(S)$ is contained in complex coordinate hyperplanes. Clearly, $\mathbb{C}^n \setminus S$ is biholomorphic to $\mathbb{C}^n \setminus F(S)$ which contains $(\mathbb{C}^*)^n$ as a dense subset, hence is a toric manifold. This applies in particular, to the complement of a tame set in the sense of Rosay-Rudin [37] in \mathbb{C}^n , $n \geq 2$.

Example 5.9. Quotients of toric manifolds by discrete groups of automorphisms acting properly discontinuously. This includes all complex tori, the Iwasawa manifold and the Hopf manifolds.

Example 5.11. Calabi-Eckmann manifolds. In fact, we know from [5] that every Calabi-Eckmann manifold $M_{m,n}$ is a complex manifold homeomorphic to the Cartesian product $\mathbb{S}^{2m+1} \times \mathbb{S}^{2n+1}$ of two odd-dimensional spheres, and one can choose a cover of coordinate domains $V_{\alpha\beta}$ ($\alpha = 0, \dots, m; \beta = 0, 1, \dots, n$) defined by

$$V_{\alpha\beta} = \{(z, z') \in \mathbb{S}^{2m+1} \times \mathbb{S}^{2n+1} \subset \mathbb{C}^{m+1} \times \mathbb{C}^{n+1} : z_\alpha z'_\beta \neq 0\},$$

which is homeomorphic to $\mathbb{C}^{m+n} \times \mathbb{T}^1$ where \mathbb{T}^1 is a 1-dimensional complex torus, so that the complex structure of $\mathbb{C}^{m+n} \times \mathbb{T}^1$ gives local coordinates of $V_{\alpha\beta}$. It follows immediately that $M_{m,n}$ is universally dominated.

Example 5.12. Let M be a universally dominated manifold and f a holomorphic function on M . Then the complement of the graph of f in $M \times \mathbb{C}$ is universally dominated. Indeed, it is biholomorphic to $M \times \mathbb{C}^*$ via the automorphism $(\text{id}_M, \text{id}_{\mathbb{C}} - f)$ of $M \times \mathbb{C}$. It is unknown whether the condition can be weakened to that f is only meromorphic. The special case when $M = \mathbb{C}$ has been verified by Buzzard-Lu (cf. [4], Theorem 5.2).

Example 5.13. The complement M of $k \leq n + 1$ distinct hyperplanes in general position in \mathbb{P}^n is universally dominated. To see this, simply take a complex chart $(\mathbb{C}^n; z)$ in \mathbb{P}^n so that the restriction of M to this complex chart is $(\mathbb{C}^*)^{k-1} \times \mathbb{C}^{n-k+1}$, which is dense in \mathbb{P}^n . On the other side, the image of any holomorphic map from \mathbb{C} to the complement of $n + 2$ distinct hyperplanes in \mathbb{P}^n lies in a hyperplane (cf. [20]), thus the latter is not universally dominated.

In general, it is very difficult to determine whether the complement of a given divisor S in a non-hyperbolic manifold is universally dominated (e.g., Kobayashi's conjecture). Below we present a useful method of constructing divisors with universally dominated complements in \mathbb{C}^n or \mathbb{P}^n . Put

$$\mathcal{F}_n := \{f \in \mathcal{O}(\mathbb{C}^n) : \mathbb{C}^n \setminus Z_f \text{ is universally dominated}\}$$

where $Z_f = \{f = 0\}$. Clearly, $z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in \mathcal{F}_n$ for any nonnegative integers $\alpha_1, \dots, \alpha_n$. Put

$$S = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_{n-1})z_n + g(z_1, \dots, z_{n-1}) = 0\}$$

where $f \in \mathcal{F}_{n-1}$ and $g \in \mathcal{O}(\mathbb{C}^{n-1})$. We claim that $\mathbb{C}^n \setminus S$ is universally dominated. To see this, simply view $((\mathbb{C}^{n-1} \setminus Z_f) \times \mathbb{C}) \cap S$ as the graph of the holomorphic function

$$h := -g/f$$

on $\mathbb{C}^{n-1} \setminus Z_f$ so that its complement in $(\mathbb{C}^{n-1} \setminus Z_f) \times \mathbb{C}$ is universally dominated by Example 5.12, so is $\mathbb{C}^n \setminus S$.

It follows immediately that $\mathbb{P}^n \setminus \hat{S}$ is universally dominated if \hat{S} is defined by

$$z_1^{\alpha_1} \cdots z_n^{\alpha_n} z_0 + \sum_{\gamma_1 + \dots + \gamma_n = \alpha_1 + \dots + \alpha_n + 1} c_{\gamma_1 \dots \gamma_n} z_1^{\gamma_1} \cdots z_n^{\gamma_n} = 0$$

where $\alpha_1, \dots, \alpha_n$ are nonnegative integers. As a consequence, we get

Proposition 5.14. (1) *The complement of any smooth quadric hypersurface in \mathbb{P}^n is universally dominated.*

(2) *The complement of the universal hypersurface of degree d in $\mathbb{P}^m \times \mathbb{P}^n$ with $m = \binom{n+d}{n} - 1$ is universally dominated.*

Proof. (1) A quadric hypersurface in \mathbb{P}^n is defined by a homogeneous polynomial of degree 2. After a change of coordinates, we may assume that hypersurface is

$$Q_k := \{(z_0 : z_1 : \dots : z_n) : z_0^2 + z_1^2 + \dots + z_k^2 = 0\}$$

where k is the rank of Q_k . Clearly, Q_k is smooth if and only if $k = n$. We claim that $\mathbb{P}^n \setminus Q_n$ is universally dominated. To see this, take first a change of coordinates as follows

$$z_0 = \zeta_0 + i\zeta_1, z_1 = \zeta_1 + i\zeta_2, \dots, z_n = \zeta_n + i\zeta_0,$$

so that the equation becomes

$$\zeta_0\zeta_1 + \zeta_1\zeta_2 + \dots + \zeta_n\zeta_0 = 0.$$

Next choose a new coordinate system:

$$t_0 = \zeta_0, \dots, t_{n-1} = \zeta_{n-1}, t_n = \zeta_1 + \zeta_n,$$

so that

$$Q_n = \{(t_0 : t_1 : \dots : t_n) : t_0 t_n + \text{polynomial of } (t_1, \dots, t_n) = 0\}.$$

Thus $\mathbb{P}^n \setminus Q_n$ is universally dominated.

(2) Recall that the universal hypersurface $\mathcal{S}_{n,d}$ of degree d in \mathbb{P}^n is defined by

$$\sum_{\alpha_0 + \dots + \alpha_n = d} t_{\alpha_0 \dots \alpha_n} z_0^{\alpha_0} \cdots z_n^{\alpha_n} = 0$$

where $(z_0 : z_1 : \cdots : z_n)$ is the homogeneous coordinate of \mathbb{P}^n and $(t_{\alpha_0 \cdots \alpha_n})_{\alpha_0 + \cdots + \alpha_n = d}$ is the homogeneous coordinate of \mathbb{P}^m . It is easy to see that $(\mathbb{P}^m \times \mathbb{P}^n) \setminus S_{n,d}$ is universally dominated. Q.E.D.

Now recall the following

Green's Theorem (cf. [21]). *Let $\mathbb{F}_{n,d}$ be the Fermat variety of degree d in \mathbb{P}^n :*

$$z_0^d + z_1^d + \cdots + z_n^d = 0.$$

Then the following conclusions hold:

- (i) *If $F : \mathbb{C}^m \rightarrow \mathbb{P}^n$ is a holomorphic map with image lying in $\mathbb{F}_{n,d}$ with $d > n^2 - 1$, then the image of F lies in a linear subspace of dimension $\left\lfloor \frac{n-1}{2} \right\rfloor$.*
- (ii) *If $F : \mathbb{C}^m \rightarrow \mathbb{P}^n$ is a holomorphic map whose image omits $\mathbb{F}_{n,d}$ with $d > n(n+1)$, then the image of F lies in a linear subspace of dimension $\left\lfloor \frac{n}{2} \right\rfloor$.*

Proposition 5.15. *There exists a complex analytic family of $(n+1)$ -dimensional quasiprojective manifolds M_t , $t \in \mathbb{C}$, such that M_t is universally dominated for all but $t = 0$.*

Proof. By virtue of Green's theorem, we may choose $d > n(n+1)$ so that $\mathbb{P}^n \setminus \mathbb{F}_{n,d}$ is not universally dominated. Put $U_0 = \{z_0 \neq 0\}$. Clearly, $U_0 \setminus \mathbb{F}_{n,d}$ is not universally dominated, for it is dense in $\mathbb{P}^n \setminus \mathbb{F}_{n,d}$. Note also that

$$\mathbb{F}_{n,d} \cap U_0 = \{(1 : z_1 : \cdots : z_n) : z_1^d + \cdots + z_n^d = -1\}.$$

Put $S_0 = (\mathbb{F}_{n,d} \cap U_0) \times \mathbb{C}$. Identifying U_0 with \mathbb{C}^n , we conclude that $\mathbb{C}^{n+1} \setminus S_0 = (U_0 \setminus \mathbb{F}_{n,d}) \times \mathbb{C}$ is not universally dominated by virtue of Proposition 5.3/(1). Now construct a complex analytic family of hypersurfaces S_t in \mathbb{C}^{n+1} as follows

$$z_1^d + \cdots + z_n^d = -1 + tz_{n+1}, \quad t \in \mathbb{C}.$$

Put $M_t = \mathbb{C}^{n+1} \setminus S_t$. Clearly, M_t is universally dominated if and only if $t \neq 0$. Q.E.D.

The central property of universally dominated spaces is the following

Theorem 5.16. *Let M_1 be a universally dominated complex space and M_2 a complex manifold. Suppose there exists a surjective meromorphic map $\Phi : M_1 \rightarrow M_2$, then M_2 is also universally dominated.*

Proof. The argument is inspired by Kobayashi [30], Lemma 3.5.29. Take first a dominant morphism $F : \mathbb{C} \rightarrow M_1$. Let S be the singular set of Φ . Clearly, $\Phi \circ F : \mathbb{C} \rightarrow M_2$ is a meromorphic map with singularity set $F^{-1}(S)$. Since $\overline{F(\mathbb{C})} = M_1$ and Φ is surjective, $F^{-1}(S) \neq \mathbb{C}$. Now M_2 is nonsingular and $\dim \mathbb{C} = 1$, we conclude that the singularities of $\Phi \circ F$ are all removable, i.e., $\Phi \circ F$ is actually holomorphic. Since there is an open dense subset $U \subset M_1$ so that $\Phi : U \rightarrow M_2$ is a dominant morphism, we see that $\Phi \circ F : \mathbb{C} \rightarrow M_2$ is also a dominant morphism. Q.E.D.

Corollary 5.17. (1) *Let M_1 be a complex manifold bimeromorphically equivalent to a complex space M_2 . If M_2 is universally dominated, so is M_1 .*

(2) *Universally dominatedness is stable under birational transformations.*

Proof. (1) Let $\Phi : M_1 \rightarrow M_2$ be a bimeromorphic map. It suffices to apply the previous theorem to $\Phi^{-1} : M_2 \rightarrow M_1$.

(2) Let $F : M_1 \rightarrow M_2$ be a birational map between algebraic varieties M_1, M_2 . Suppose M_2 is universally dominated. Let $\pi : \widetilde{M}_1 \rightarrow M_1$ be a desingularization of M_1 . Clearly, \widetilde{M}_1 is birationally equivalent to M_2 , thus it is also universally dominated by virtue of the previous conclusion (1). Since π is a surjective holomorphic map, we conclude that M_1 is universally dominated. Q.E.D.

Example 5.18. Unirational varieties and Kummer manifolds are universally dominated. Indeed, if M is unirational, then there exists a surjective rational map $F : \mathbb{P}^n \rightarrow M$. Let $\pi : \widetilde{M} \rightarrow M$ be a desingularization of M . Clearly, $\pi^{-1} \circ F : \mathbb{P}^n \rightarrow \widetilde{M}$ is a surjective rational map. Thanks to Theorem 5.16, \widetilde{M} is universally dominated, so is M (see also [41] for a different approach). For a Kummer manifold M , there exist an abelian variety A and a finite group G of holomorphic automorphisms of A such that M is bimeromorphically equivalent to the quotient variety A/G , thus it has to be universally dominated by virtue of Corollary 5.17/(1).

We have known from Proposition 5.2/(1) that universally dominatedness implies the Kobayashi pseudodistance $k_M \equiv 0$, the converse fails, however:

Proposition 5.19. *There is a complex space M which is not universally dominated but $k_M \equiv 0$.*

Proof. We start with a hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ and a point $p \in \mathbb{P}^n \setminus \mathbb{P}^{n-1}$. Fix a nonsingular curve $C \subset \mathbb{P}^{n-1}$ of genus ≥ 2 . The cone over C with vertex p is defined as

$$\text{cone}(C, p) = \bigcup_{q \in C} \overline{qp},$$

i.e., the union of the complex lines jointing p to points of C . Since $\text{cone}(C, p)$ is a union of \mathbb{P}^1 intersects at p , its Kobayashi pseudodistance vanishes identically (compare [30], Example 3.2.21). On the other hand, since the projection $\pi_p : \text{cone}(C, p) \rightarrow C$ is a surjective rational map, $\text{cone}(C, p)$ cannot be universally dominated by virtue of Theorem 5.16. Q.E.D.

We end this section by proposing a few open problems.

Problem 5.20. Suppose M is a universally dominated manifold and \widetilde{M} is an unramified holomorphic covering of M . Is \widetilde{M} universally dominated?

Problem 5.21. Let M be a holomorphic fiber bundle. Suppose both the base and the fiber are universally dominated. Is M also universally dominated?

We give a partial answer as follows:

Proposition 5.22. *Let M be a universally dominated manifold and E a holomorphic principle G -bundle over M with G being a universally dominated connected complex Lie group. Then E is also universally dominated.*

Proof. Take a dominant morphism $F : \mathbb{C} \rightarrow M$. The pullback bundle, $F^*(E)$ over \mathbb{C} is holomorphically trivial, thanks to Grauert's Oka-principle (cf. [18]). It follows that $F^*(E)$ is universally dominated, so is E . Q.E.D.

As an application, we conclude that the Stiefel manifold $\text{St}(k, n)$, i.e., the set of all k -tuples of linearly independent vectors in \mathbb{C}^n , is universally dominated. To see this, simply view $\text{St}(k, n)$ as a principle fiber bundle over the Grassmannian $\mathbb{G}(k, n)$, with fiber $\text{GL}(k, \mathbb{C})$.

Problem 5.23. Is every projective algebraic manifold of general type *not* universally dominated?

Of course, a resolution of the celebrated Green-Griffiths conjecture (cf. [22]) would give a positive answer to this problem.

Problem 5.24. Suppose M is a universally dominated manifold of dimension n and S is a closed subset in M so that $M \setminus S$ is connected. Under which condition is $M \setminus S$ universally dominated?

For instance, we do not know whether the complement M of the totally real plane

$$\{(z_1, z_2) : \text{Re } z_1 = \text{Re } z_2 = 0\}$$

in \mathbb{C}^2 is universally dominated. It is known that M contains a Fatou-Bieberbach domain (cf. [37], Example 9.6).

Many examples of universally dominated manifolds given in this paper are also elliptic in the sense of Gromov [24] or Oka in the sense of Forstnerič (see [13] for the definition). It would be interesting to study the relationships between these manifolds.

Remarks. a) Recently, Forstnerič and Ritter [16] observed that every Oka manifold is dominated by \mathbb{C} , and hence is universally dominated in view of Proposition 5.1. For a list of examples of Oka manifolds, we refer to [13] or [15]. In particular, every complex Lie group is Oka. Forstnerič-Ritter also proved that the complement of a compact polynomially convex set in \mathbb{C}^n for $n \geq 2$ is universally dominated. On the other hand, it is known that the complement of the unit closed ball in \mathbb{C}^n for $n \geq 3$ fails to be elliptic (or subelliptic) (cf. [2]).

b) Forstnerič and Larusson [15] constructed a complex analytic family of *compact* complex surfaces such that the central fibre is an Inoue-Hirzebruch surface which is not universally dominated since its universal covering possesses a nonconstant negative psh function, continuous outside of a curve, whereas all other fibres are minimal Enoki surfaces which are Oka, so they are universally dominated. We thank Finnur Larusson for pointing out this fact.

6 A remark on manifolds with nonconstant bounded holomorphic functions

Usually it is very difficult to know whether there exist nonconstant bounded holomorphic functions on a given complex manifold. Nevertheless, we have the following

Proposition 6.1. *Let M be a complex manifold which admits a nonconstant meromorphic function. Then there is an analytic hypersurface S in M such that the universal covering of $M \setminus S$ admits nonconstant bounded holomorphic functions. In particular, $M \setminus S$ is not universally dominated.*

We need the following

Picard's Little Theorem. *Let X be a complex manifold whose universal covering \tilde{X} admits no nonconstant bounded holomorphic functions. Then every nonconstant holomorphic function on X omits at most one value.*

Proof. Suppose there is a nonconstant holomorphic function $f : X \rightarrow \mathbb{C} \setminus \{a, b\}$. Let \tilde{f} be a lift of f to the universal coverings of X and $\mathbb{C} \setminus \{a, b\}$, i.e., \tilde{X} and \mathbb{D} . Then it is a nonconstant bounded holomorphic function on \tilde{X} . Contradiction. Q.E.D.

Although the proof is trivial, this result still has several amusing consequences. For instance, it follows that *every* nonconstant holomorphic function on a (Stein) quotient of \mathbb{C}^n by a free, properly discontinuous group of automorphisms omits at most one value.

Proof of Proposition 6.1. We fix a nonconstant meromorphic function f on M and choose a cover $\{U_j\}$ of M such that

$$f = g_j/h_j$$

on U_j , where g_j, h_j are two relatively prime holomorphic functions. Thus

$$f_{jk} = g_j/g_k = h_j/h_k$$

is a non-vanishing holomorphic function on $U_j \cap U_k$. Thus the set

$$S_0 := \bigcup \{z \in U_j : h_j(z) = 0\}$$

is an analytic hypersurface of M and f is a nonconstant holomorphic function on $M \setminus S_0$. Fix any two distinct complex numbers a, b and put $S := S_0 \cup f^{-1}(a) \cup f^{-1}(b)$. Thanks to Picard's Little Theorem, we see that the universal covering of $M \setminus S$ admits nonconstant bounded holomorphic functions. We claim that $M \setminus S$ is not universally dominated. Suppose on the contrary there is a dominant morphism $F : \mathbb{C} \rightarrow M \setminus S$. Let \tilde{F} be a lift of F to the universal covering X of $M \setminus S$ and h a nonconstant bounded holomorphic function on X . Then we get a nonconstant bounded holomorphic function $h \circ \tilde{F}$ on \mathbb{C} , contradicts with Liouville's theorem. Q.E.D.

Acknowledgements. We would like to thank Professor Franc Forstnerič for numerous comments on this paper. We also thank Dr. Qi'an Guan for catching an inaccuracy in the proof of Theorem 1.3.

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